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Calculations for the Greenhouse Development Rights Calculator

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Abstract

The Greenhouse Development Rights (GDRs) on-line calculator is a web-based tool for examining the implications of different assumptions under the GDRs framework for allocating the costs of climate mitigation and adaptation. This working paper documents the main GDRs-relevant calculations, in particular those depending on within-country income distributions. The paper also presents the derivation of the Gini coefficient for a weighted sum of lognormal distributions. This is a general result that can be used to calculate regional Gini coefficients in applications other than the GDRs calculator.

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Introduction

The Greenhouse Development Rights (GDRs) Framework (Baer et al., 2008) is a general framework for burden-sharing for climate change that takes both responsibility and financial capacity into account. The framework takes the individual as the unit of analysis, which means that within-country income distributions are a necessary input into calculations that make use of the framework. The availability, since the mid-1990s, of a reasonably complete international database on income distribution (Deininger and Squire, 1996; UNU-WIDER, 2008) makes this calculation possible. While income mobility – the trajectory of individuals over time within an income distribution – would also be useful, there are insufficient data on income mobility to support an international calculation.

An on-line calculator is available that is consistent with the GDRs framework and that is used by the developers of the GDRs framework in order to explore the implications of the framework at national and individual level.[1] This working paper describes some of the calculations that are implemented in the calculator. The calculator "engine" itself is available as a CGI (web-enabled) program written in C. The code is open-source, and can be downloaded or browsed online.[2]

Lognormal Income Distributions

Following Kemp-Benedict (2001) and Lopez and Servén (2006), income is assumed to be distributed lognormally within countries. The lognormal distribution has two parameters: the mean income $\bar{y}$ and the standard deviation of the log of income, σ. The standard deviation of the log of income is a measure of how equally or unequally distributed income is within the country, and can be related to the well-known Gini coefficient G using the following formula,

$$\sigma = \sqrt{2}\, N^{-1}\left(\frac{1+G}{2}\right). \tag{1}$$

In this expression, N^{-1} is the inverse normal function.

It is convenient when working with the distribution within an individual country to transform income y into a new variable z, given by

$$z = \frac{1}{\sigma}\ln(y/\bar{y}) + \frac{\sigma}{2}. \tag{2}$$

In terms of z, the lognormal becomes simply a normal distribution,

$$y \sim \text{Lognormal}(\bar{y}, \sigma) \;\Rightarrow\; z \sim N(0,1). \tag{3}$$

"Partial Moments" of the lognormal income distribution

The GDRs framework proposes that income below a certain lower income threshold y_l is exempt, in that it does not contribute to an individual's capacity to pay. Furthermore, in the practical implementation of the framework within the calculator, it is assumed that emissions within countries vary with income with an elasticity that is the same for all countries, and emissions associated with incomes below y_l are exempt, in that they do not contribute to an individual's responsibility.

1 The calculator can be accessed at http://www.gdrights.org/calculator/.

2 To download or browse the code, go to http://gdrs.sourceforge.net/.

More specifically, individual per capita emissions are assumed to vary in the following way with income,

$$\epsilon_i(y) = A_i y^{\gamma} , \quad (4)$$

where $\epsilon_i(y)$ is emissions per capita for individuals in a narrow range of incomes around y for country i, and A_i is a country-specific constant. The elasticity γ is the same for all countries. Chakravarty et al. (2009) report evidence that this assumption stands up reasonably well to empirical test, and that the data are consistent with γ between about 0.7 and 1.0.

In order to implement the framework, it is necessary to evaluate integrals of the following form, which are referred to in this working paper as "partial moments",

$$M_{\gamma}(y; y_l, \sigma, \bar{y}) \equiv \int_{y_l}^{\infty} dy \, \left(y^{\gamma} - y_l^{\gamma}\right) f(y; \bar{y}, \sigma) , \quad (5)$$

where $f(y; \bar{y}, \sigma)$ is the (lognormal) income distribution. Using the convenient change of variables given in Equation (2), for a lognormal distribution the partial moments can be shown to be equal to

$$M_{\gamma}(y; y_l, \sigma, \bar{y}) = \bar{y}^{\gamma} e^{\frac{\sigma^2}{2}\gamma(\gamma-1)} \left(1 - N(z_l - \gamma\sigma)\right) - y_l^{\gamma}\left(1 - N(z_l)\right) , \quad (6)$$

where N is the cumulative normal distribution. This follows from the following equation,

$$\int_{y_l}^{\infty} dy \; y^{\gamma} f(y; \sigma, \bar{y}) \; = \; \bar{y}^{\gamma} e^{-\gamma\sigma^2/2} \int_{z_l}^{\infty} dz \; e^{\gamma\sigma z} \frac{1}{\sqrt{2\pi}} e^{-\frac{1}{2}z^2} \; = \; \bar{y}^{\gamma} e^{\frac{\sigma^2}{2}\gamma(\gamma-1)} \left(1 - N(z_l - \gamma\sigma)\right) . \quad (7)$$

Capacity and Responsibility

In principle it is possible to have more than one income threshold and have different fractions of income contribute toward capacity to different degrees at different income levels. In some versions of the calculator this is implemented using two different thresholds, a lower threshold y_l, and an upper threshold y_u. Income is exempt below the lower threshold, while above the upper threshold, 100% of income contributes to capacity. Between the two thresholds, a fraction φ of each marginal increment of income contributes toward capacity. The definitions of capacity and responsibility will be given for this general case.

Individual capacity and responsibility per capita

Individual capacity per capita in country i, $c_i(y)$, is given by

$$c_i(y) = \begin{cases} 0 & y < y_l \\ \varphi(y - y_l) & y_l \leqslant y \leqslant y_u \\ \varphi(y_u - y_l) + (y - y_u) & y > y_u \end{cases} . \quad (8)$$

Responsibility is based on emissions above the threshold, where individual emissions per capita vary as in Equation (4). The annual contribution to individual per capita responsibility r_i^{ann} is then given by

$$r_i^{\text{ann}}(y)=\begin{cases} 0 & y<y_l \\ \varphi A_i(y^\gamma - y_l^\gamma) & y_l \leqslant y \leqslant y_u \\ A_i\left[\varphi(y_u^\gamma - y_l^\gamma)+(y^\gamma - y_u^\gamma)\right] & y>y_u \end{cases}. \tag{9}$$

The coefficient A_i in Equation (9) can be determined from the total national annual emissions E_i, using the relationship

$$E_i = \int_0^\infty dy\ A_i y^\gamma f(y;\bar{y}_i,\sigma_i) = A_i \bar{y}_i^\gamma e^{\frac{\sigma_i^2}{2}\gamma(\gamma-1)}. \tag{10}$$

From this equation it can be seen that

$$A_i = \frac{E_i}{\bar{y}_i^\gamma} e^{-\frac{\sigma_i^2}{2}\gamma(\gamma-1)}, \tag{11}$$

and

$$r_i^{\text{ann}}(y)=\begin{cases} 0 & y<y_l \\ \frac{E_i}{\bar{y}_i^\gamma} e^{-\frac{\sigma_i^2}{2}\gamma(\gamma-1)} \varphi(y^\gamma - y_l^\gamma) & y_l \leqslant y \leqslant y_u \\ \frac{E_i}{\bar{y}_i^\gamma} e^{-\frac{\sigma_i^2}{2}\gamma(\gamma-1)}\left[\varphi(y_u^\gamma - y_l^\gamma)+(y^\gamma - y_u^\gamma)\right] & y>y_u \end{cases}. \tag{12}$$

NATIONAL CAPACITY AND RESPONSIBILITY

National capacity C_i is given by integrating per capita capacity [Equation (8)] over the income distribution. Using the formula for partial moments given in Equation (6), and rearranging, this can be shown to be equal to

$$C_i = \bar{y}\left[1-(1-\varphi)N(z_u-\sigma_i)-\varphi N(z_l-\sigma_i)\right]-(1-\varphi)y_u\left(1-N(z_u)\right)-y_l\left(1-N(z_l)\right). \tag{13}$$

Note that in the case $\varphi = 1$, this collapses to the expression for a single threshold,

$$\varphi=1 \ \Rightarrow\ C_i=\bar{y}\left[1-N(z_l-\sigma_i)\right]-y_l\left(1-N(z_l)\right). \tag{14}$$

National annual responsibility R_i^{ann} is found by integrating per capita annual responsibility [Equation (12)] over the income distribution. This can be shown to equal

$$\begin{aligned} R_i^{\text{ann}} = {} & E_i\left[1-(1-\varphi)N(z_u-\gamma\sigma_i)-\varphi N(z_l-\gamma\sigma_i)\right] \\ & -E_i e^{-\frac{\sigma^2}{2}\gamma(\gamma-1)}\left\{(1-\varphi)\left(\frac{y_u}{\bar{y}}\right)^\gamma\left(1-N(z_u)\right)+\left(\frac{y_l}{\bar{y}}\right)^\gamma\left(1-N(z_l)\right)\right\}. \end{aligned} \tag{15}$$

As with capacity, this collapses to the one-threshold case when $\varphi = 1$.

In any given year Y, national responsibility R_i is calculated as cumulative annual responsibility from a starting year t_{ref}. Adding a time index t to annual responsibility, this can be written

$$R_{i,Y} = \sum_{t=t_{\mathrm{ref}}}^{Y} R_{i,t}^{\mathrm{ann}} \,. \tag{16}$$

In subsequent formulas, the time index will be suppressed.

The Responsibility-Capacity Indicator

The Responsibility-Capacity Indicator (RCI) is the key indicator for the GDRs framework. It is used to allocate burden-sharing. National RCI, as implemented in the calculator, is a weighted sum of the national share of global responsibility and capacity. That is,

$$\mathrm{RCI}_i = a \frac{R_i}{\sum_{i=1}^{N} R_i} + (1-a) \frac{C_i}{\sum_{i=1}^{N} C_i} \,, \tag{17}$$

where N is the number of countries and a lies between zero and one. National RCIs sum to one across countries, and so national RCIs are also the national share of global total RCI. For an individual within country i, RCI per capita, $\mathrm{rci}_i(y)$, is calculated as

$$\mathrm{rci}_i(y) = a \frac{R_i}{\sum_{i=1}^{N} R_i} \left(\frac{r_i^{\mathrm{ann}}(y)}{R_i^{\mathrm{ann}}} \right) + (1-a) \frac{c_i(y)}{\sum_{i=1}^{N} C_i} \,. \tag{18}$$

This formulation assumes that cumulative responsibility is allocated according to the current annual allocation in any given year. This is not ideal, since in fact individuals will have their own income trajectories that can be quite divergent, depending on the degree of income mobility within a country. However, given the very limited data on income mobility it is not feasible to try to capture this dynamic.

Gini Coefficients for Weighted Sums of Lognormal Distributions

When preparing data for the GDRs calculator, values for China are calculated using data on income distributions and populations for Hong Kong and Mainland China. The combined Gini coefficient for China is estimated as the Gini coefficient for a weighted sum of lognormal distributions, one for Hong Kong and one for Mainland China. The general result for the Gini coefficient of a weighted sum of lognormals is presented in this section.

Statement of the result

Suppose that a region is composed of N countries, labeled by $i = 1, \ldots, N$. Each of the N countries is assumed to have lognormally distributed income, with income distributions $f_{\ln}(y, \bar{y}_i, \sigma_i)$, where $f_{\ln}(\cdot)$ is the lognormal distribution, y is per capita income, $\bar{y}_i$ is mean per capita income for country i, and σ_i is the square root of the variance of the logarithm of income. Furthermore, suppose that the share of regional population within each country is s_i, where $\sum_{i=1}^{N} s_i = 1$. Then the regional income distribution $f_{\mathrm{reg}}(y)$ is

$$f_{\mathrm{reg}}(y) = \sum_{i=1}^{N} s_i f_{\ln}(y, \bar{y}_i, \sigma_i) \,, \tag{19}$$

regional mean income is,

$$\bar{y}=\sum_{i=1}^{N} s_i \bar{y}_i \,, \tag{20}$$

and the regional Gini coefficient, G, is

$$G=1-\frac{2}{\bar{y}}\sum_{i=1}^{N}\sum_{j=1}^{N} s_i s_j I_{ij} \,, \tag{21}$$

where

$$I_{ij}=\bar{y}_j N\left[\frac{1}{\sqrt{\sigma_i^2+\sigma_j^2}}\left(\ln\frac{\bar{y}_i}{\bar{y}_j}+\frac{1}{2}\sigma_i^2-\frac{3}{2}\sigma_j^2\right)\right], \tag{22}$$

and $N(\cdot)$ is the cumulative standard normal distribution.

Derivation of the result

This section presents a derivation of Equations (21-22). The derivation starts with a general expression for the Gini coefficient as a functional of the income distribution.

Gini coefficient in terms of the income distribution

The Gini coefficient is defined in terms of the Lorenz curve, which is a plot of cumulative income against cumulative population, where the population is ranked in order of increasing income. If the Lorenz curve is represented by $L(x)$, where x is the share of cumulative population, then the Gini coefficient is

$$G=1-2\int_0^1 L(x)dx \,. \tag{23}$$

The integral in Equation (23) can be evaluated by carrying out a change of variables, from cumulative population x to income y. They are related through the income distribution. For a generic income distribution $f(y)$,

$$x=\int_0^y dy'\ f(y') \,. \tag{24}$$

Cumulative income – that is, the Lorenz curve – can then be calculated as

$$L(x)=\frac{1}{\bar{y}}\int_0^{y(x)} dy'\ y' f(y') \,. \tag{25}$$

where the dependence of the upper limit $y(x)$ of the integral through Equation (24) is shown explicitly. By changing to y as the variable in the integrand, Equation (23) can be seen to be equivalent to

$$G=1-\frac{2}{\bar{y}}\int_0^\infty dy\ f(y)\int_0^y dy'\ y' f(y') \,. \tag{26}$$

This is a general result, for any income distribution $f(y)$. In the next section it is applied to the regional income distribution given in Equation (19).

Gini coefficient for a weighted sum of lognormal distributions

Applying Equation (26) to the regional income distribution in Equation (19) gives

$$G=1-\frac{2}{\bar{y}}\sum_{i=1}^{N}\sum_{j=1}^{N} s_i s_j I_{ij}\,, \tag{27}$$

that is, Equation (21), where

$$I_{ij}=\int_0^\infty dy\ f_{\ln}(y,\bar{y}_i,\sigma_i)\int_0^y dy'\ y' f_{\ln}(y',\bar{y}_j,\sigma_j)\,. \tag{28}$$

The double integral in Equation (7) is difficult to evaluate as it is written. It can be transformed using the following procedure. First, define

$$\hat{I}_{ij}(a)=\int_0^\infty dy\ f_{\ln}(y,\bar{y}_i,\sigma_i)\int_0^{ay} dy'\ y' f_{\ln}(y',\bar{y}_j,\sigma_j)\,. \tag{29}$$

This function is almost identical to I_{ij}, except that a new variable a multiplies the upper limit on the inner integral over y'. This function has the properties that

$$\hat{I}_{ij}(0)=0 \ \text{ and } \ \hat{I}_{ij}(1)=I_{ij}\,. \tag{30}$$

Taking the derivative of $\hat{I}_{ij}(a)$ with respect to a reduces Equation (29) to a single integral, but with the consequence that it must later be integrated over a to recover the full function. This turns out to be simpler than evaluating Equation (7) directly. The derivative of $\hat{I}_{ij}(a)$ with respect to a is

$$\frac{d\,\hat{I}_{ij}(a)}{d\,a}=\int_0^\infty dy\ a\,y^2 f_{\ln}(y,\bar{y}_i,\sigma_i)\,f_{\ln}(ay,\bar{y}_j,\sigma_j)\,. \tag{31}$$

This integral is then evaluated by using the standard form for the lognormal income distribution,

$$f_{\ln}(y,\bar{y},\sigma)=\frac{1}{\sqrt{2\pi}\,\sigma\,y}\,e^{-\frac{1}{2\sigma^2}\left(\ln\frac{y}{\bar{y}}+\frac{\sigma^2}{2}\right)^2}\,. \tag{32}$$

Substituting Equation (32) in Equation (31), a change of variables from y to $\ln y$ in the integral transforms it into an integral over normal distributions.

Integrals of products of normal distributions can be carried out using standard techniques. The calculation is lengthy, but straightforward, and produces the following result:

$$\frac{d\,\hat{I}_{ij}(a)}{d\,a}=\frac{1}{\sqrt{2\pi\left(\sigma_i^2+\sigma_j^2\right)}}\,e^{-\frac{1}{2}\frac{1}{\sigma_i^2+\sigma_j^2}\left[\left(\ln a+A_i-A_j\right)^2-\sigma_i^2\sigma_j^2-2A_i\sigma_j^2+2\left(\ln a-A_j\right)\sigma_i^2\right]}\,, \tag{33}$$

where

$$A_k=\ln\bar{y}_k-\frac{\sigma_k^2}{2}\,. \tag{34}$$

The expression for $\hat{I}_{ij}(a)$ itself is found by integrating the right-hand side in Equation (33) from zero to one and applying the boundary condition $\hat{I}_{ij}(0)=0$ from Equation (30). This can be done by performing a change of variables from a to $\ln a$, which changes the bounds of the integral from 0 and 1 to $-\infty$ and 0. This results in a standard integral of the normal distribution over half of its domain, and can be done exactly. The result is Equation (22).

Conclusion

This working paper presents a derivation of some of the core calculations used by the Greenhouse Development Rights (GDRs) online calculator. GDRs is a rights-based allocation framework, and takes the individual, rather than the country, as the basis for national allocations. For this reason, it is necessary to capture within-country income distributions, and this paper focuses on the calculations within the GDRs calculator that involve lognormal income distributions. The main results of the paper are an explicit calculation for the responsibility-capacity indicator (RCI) as implemented in the online calculator, and an expression for the Gini coefficient for a weighted sum of lognormal distributions.

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